



9D space – the best way to understand paleostress analysis

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Abstract: The goal of this paper is to illustrate some basic facts about paleostress analysis using C-lines in 9-dimensional space. In particular, two optimization parameters are proposed to validate the reliability of numerical solutions. The Gram determinant was used to evaluate the error of numerical outputs, and the antisymmetrical component of the C-line was used as a correctness parameter of T_{σ} solution.

Keywords: paleostress analysis, fault-slip data, 9D space, C-line, principal stresses, inverse method.

Paleostress analysis of fault-slip data sets is an important method for reconstructing tectonic events in deformed rocks. Several approaches based on an inverse technique were proposed in the past, but Fry's idea (Fry, 1999, 2001) led to a breakthrough (was a turning point) in its visualization. His theory was expanded and adapted by Shan *et al.* (2003), Sato and Yamaji (2006), among others. This paper highlights some new aspects of this method in full 9D space, which contribute to a better understanding of paleostress problems.

Linear algebraic expressions will be needed for our explanation, so it is necessary to outline the notation system first. The bold characters note columnar vectors (from 3 to 9 dimensions) and angle brackets note square matrices (from 3 to 9 dimensions). Italics indicate scalars.

Principles

Fault-slip data are the main data for paleostress analysis. Each fault-slip datum consists of the orientation of the fault surface (dip direction α and φ dip) and the direction of striation complemented by sense of slip (expressed as angle of pitch p). They can be represented by two orthogonal unit vectors

(\mathbf{n} , \mathbf{l}) and by a derived third orthogonal unit vector (\mathbf{m}) in three-dimensional real space. Vector $\mathbf{n} = (n_x; n_y; n_z)^T$ is the normal to the fault surface and is down directed (i.e. $n_z \geq 0$), vector $\mathbf{l} = (l_x; l_y; l_z)^T$ is a vector oriented in the direction of hanging-wall movement (i.e. parallel to striation). The third vector, $\mathbf{m} = (m_x; m_y; m_z)$, lies in the fault surface at a right angle to the striation, which means it is perpendicular to \mathbf{n} and \mathbf{l} (Fig. 1). It could be counted as the vector product $\mathbf{m} = \mathbf{n} \times \mathbf{l}$.

To solve the inverse problem, Fry (1999, 2001) proposed to use the fact that shear stress S_m in \mathbf{m} direction is equal to zero:

$$S_m = \mathbf{m} \cdot [T_{\sigma}] \cdot \mathbf{n} = \sum_{i,j} m_i n_j \sigma_{ij} = 0 \quad (1)$$

where $[T_{\sigma}]$ is the 3-dimensional matrix of the stress tensor:

$$[T_{\sigma}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad (2)$$

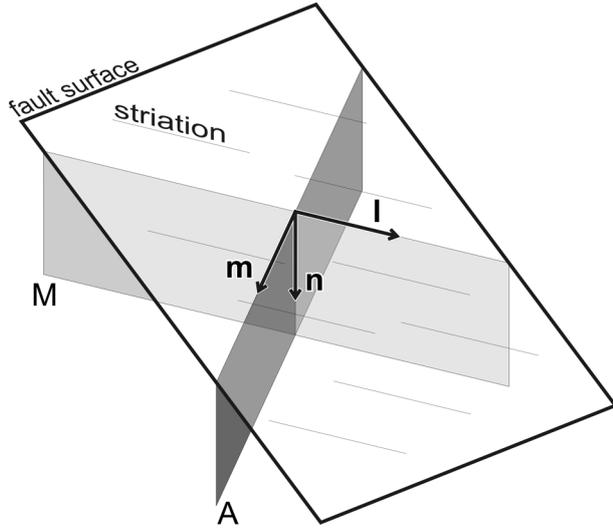


Figure 1. A visualization of three unit vectors representing a fault-slip datum of sinistral-normal fault. Vector \mathbf{n} is a down directed normal vector to the fault surface, vector \mathbf{l} is a vector oriented in the direction of hanging-wall movement (i.e. parallel to striation), and vector \mathbf{m} lies in the fault surface at right angles to the striation.

and \mathbf{m} and \mathbf{n} (i.e. m_i and n_i) are known parameters of fault-slip orientation. The sum $\sum m_i n_j \sigma_{ij}$ may be expressed as a scalar product of two vectors:

$$\sum_{i,j} m_i n_j \sigma_{ij} = \mathbf{C} \cdot \mathbf{T}_\sigma = \begin{pmatrix} m_x n_x \\ m_x n_y \\ m_x n_z \\ m_y n_x \\ m_y n_y \\ m_y n_z \\ m_z n_x \\ m_z n_y \\ m_z n_z \end{pmatrix} \cdot \begin{pmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yx} \\ \sigma_{yy} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{zy} \\ \sigma_{zz} \end{pmatrix} \quad (3),$$

where \mathbf{C} is the 9D vector representing fault-slip data (“C-line” after, Shan *et al.*, 2004; Li *et al.*, 2005) based on $m_i n_j$ parameters when $C_{ij} = m_i \cdot n_j$ and the vector \mathbf{T}_σ is a stress tensor rearranged from σ_{ij} to the 9D vector in the same way. Equation (1) can be visualized as a perpendicularity condition of the considered vectors \mathbf{C} and \mathbf{T}_σ in 9D space; in other words, we are looking for a stress vector \mathbf{T}_σ , which is perpendicular to the C-lines representing fault slips. It is a problem equivalent to fold axis analysis in real 3D space, where we are looking for a fold axis perpendicular to bedding normal lines. We use this analogy to explain multidimensional paleostress analysis.

Direct inversion of fault-slip data

From equation (1) rewritten in the form:

$$C_{xx}\sigma_{xx} + C_{xy}\sigma_{xy} + C_{xz}\sigma_{xz} + C_{yx}\sigma_{yx} + C_{yy}\sigma_{yy} + C_{yz}\sigma_{yz} + C_{zx}\sigma_{zx} + C_{zy}\sigma_{zy} + C_{zz}\sigma_{zz} = 0 \quad (4),$$

it is evident, that σ_{ij} represent nine unknowns we are looking for and C_{ij} are nine known coefficients. This equation has many solutions because there are so many unknowns, therefore other complementary restrictions were proposed (Fry, 1999):

1. Due to the moment equilibrium, the shearing components of the stress tensor are equal (e.g. $\sigma_{xy} - \sigma_{yx} = 0$; $\sigma_{xz} - \sigma_{zx} = 0$; $\sigma_{zy} - \sigma_{yz} = 0$), so there are only six independent components in practice.

2. The isotropic component of the stress tensor has no effect on fault-slip data and could be assumed equal to zero: $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0$.

Therefore we can solve a homogeneous system of linear equations in 9D, if we apply four limitative equations completed by four data equations (Eq. 4). The calculated result represents the direction of the \mathbf{T}_σ vector in 9D. Using the Frobenius norm I_F invariable we can calculate unknowns σ_{ij} . Because the Frobenius norm represents \mathbf{T}_σ vector magnitude, it is convenient to choose the vector \mathbf{T}_σ unitary:

$$I_F = \sqrt{\sigma_{xx}^2 + \sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yx}^2 + \sigma_{yy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 + \sigma_{zy}^2 + \sigma_{zz}^2} = 1 \quad (5).$$

The additional limitative equations could be expressed as scalar products of two 9D vectors like the equations representing fault-slip data: $\mathbf{A}_{xy} \cdot \mathbf{T}_\sigma = 0$; $\mathbf{A}_{xz} \cdot \mathbf{T}_\sigma = 0$; $\mathbf{A}_{yz} \cdot \mathbf{T}_\sigma = 0$ and $\mathbf{A}_{xyz} \cdot \mathbf{T}_\sigma = 0$, where \mathbf{A} are unit orthogonal vectors:

$$\mathbf{A}_{xy} = (0; 1/\sqrt{2}; 0; -1/\sqrt{2}; 0; 0; 0; 0; 0)^T \quad (6),$$

$$\mathbf{A}_{xz} = (0; 0; 1/\sqrt{2}; 0; 0; 0; -1/\sqrt{2}; 0; 0)^T \quad (7),$$

$$\mathbf{A}_{yz} = (0; 0; 0; 0; 0; 1/\sqrt{2}; 0; -1/\sqrt{2}; 0)^T \quad (8),$$

$$\mathbf{A}_{xyz} = (1/\sqrt{3}; 0; 0; 0; 1/\sqrt{3}; 0; 0; 0; 1/\sqrt{3})^T \quad (9),$$

to which the unknown \mathbf{T}_σ vector should also be perpendicular. From this perspective, we can use the analogous situation from 3D fold-axis analysis, where we are looking for the fold β -axis just as a direction perpendicular to the bedding planes’ normal vectors $\mathbf{n}_1, \mathbf{n}_2$. The fold β -axis is calculated as the vector product: $\boldsymbol{\beta} = \mathbf{n}_1 \times \mathbf{n}_2$.

Analogously, we can obtain the \mathbf{T}_σ vector from the generalized vector product in 9D (see mathematical compendia for definition, e.g. Onishchik, 2002, among others):

$$\mathbf{T}_\sigma^* = \{\mathbf{A}_{xy} \times \mathbf{A}_{xz} \times \mathbf{A}_{yz} \times \mathbf{A}_{xyz} \times \mathbf{C}_I \times \mathbf{C}_{II} \times \mathbf{C}_{III} \times \mathbf{C}_{IV}\} \quad (10).$$

Symbols \mathbf{C}_I , \mathbf{C}_{II} , \mathbf{C}_{III} , \mathbf{C}_{IV} are 9D vectors (C-lines) representing four fault-slip data.

Just as the length of the resultant β -axis depends on the deviation between vectors \mathbf{n}_1 , \mathbf{n}_2 , similarly, the length of the resultant vector $|\mathbf{T}_\sigma^*|$ depends on vector deviations (angles) between the \mathbf{C}_i and \mathbf{A} vectors. In other words, vector deviations between the \mathbf{C}_i and \mathbf{A} vectors are represented by this length, which is equivalent to the square root of the Gram determinant $G = \det(\mathbf{T}_\sigma^* \cdot \mathbf{T}_\sigma^{*\top})$. The Gram determinant is accordingly the way to evaluate errors of numerical inversion. Magnitudes of the Gram determinant approximating to zero indicate small angles between considered vectors and consequently large errors in results.

New coordinate system

To understand the geometry of the system formed by C-lines it is convenient to use a new coordinate system (primed) with respect to the four considered requirements. We can rearrange coordinates to be parallel to the \mathbf{A}_{xyz} , \mathbf{A}_{xy} , \mathbf{A}_{xz} and \mathbf{A}_{yz} vectors and free coordinates are delimited to the remaining five dimensions, where the \mathbf{T}_σ -vector should lie. The new coordinate system is defined in figures 2a and 2b. New coordinates may be counted by the transformation $\mathbf{T}'_\sigma = [\mathbf{R}] \cdot \mathbf{T}_\sigma$ where $[\mathbf{R}]$ is a matrix:

$$[\mathbf{R}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (11),$$

This rotation changes the “geographic” coordinates system into the coordinates where the four bottom components of rotated \mathbf{T}'_σ will be zero:

$$\mathbf{T}'_\sigma = \begin{pmatrix} \sigma'_1 \\ \sigma'_2 \\ \sigma'_3 \\ \sigma'_4 \\ \sigma'_5 \\ \sigma'_6 \\ \sigma'_7 \\ \sigma'_8 \\ \sigma'_9 \end{pmatrix} = \begin{pmatrix} \frac{\sigma_{xx} - \sigma_{yy}}{\sqrt{2}} \\ \sqrt{2} \cdot \sigma_{xy} \\ \sqrt{2} \cdot \sigma_{xz} \\ \sqrt{2} \cdot \sigma_{yz} \\ \frac{\sigma_{xx} - \sigma_{zz}}{\sqrt{6}} + \frac{\sigma_{yy} - \sigma_{zz}}{\sqrt{6}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (12).$$

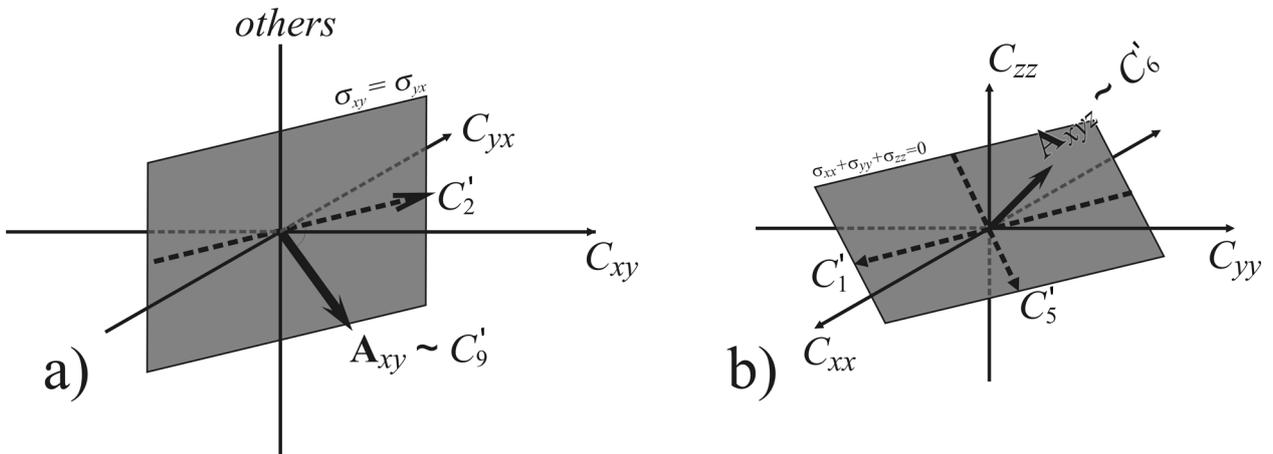


Figure 2. A definition of the new coordinate system by the normal vectors \mathbf{A}_{ij} and \mathbf{A}_{xyz} : a) the sheering components of the stress tensor are equal, e.g. $\sigma_{xy} = \sigma_{yx}$. This equation defines the hyperplane with the normal vector \mathbf{A}_{xy} which indicates the new coordinate in the turned coordinate system; b) the isotropic component of the stress tensor has no effect on fault-slip data and could be assumed equal to zero: $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0$. This equation defines the hyperplane with the normal vector \mathbf{A}_{xyz} .

As the first 6 dimensions represent the symmetrical stress tensor, we mark it as “symmetrical” subspace and the remaining 3 dimensions as “antisymmetrical” subspace.

The same transformation $\mathbf{C}' = [\mathbf{R}] \cdot \mathbf{C}$ applied to vector \mathbf{C} produces:

$$\mathbf{C}' = \begin{pmatrix} C'_1 \\ C'_2 \\ C'_3 \\ C'_4 \\ C'_5 \\ C'_6 \\ C'_7 \\ C'_8 \\ C'_9 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \cdot \sin p \cdot \sin \varphi \cdot \frac{1}{2} \sin 2\alpha & -\sqrt{2} \cdot \cos p \cdot \frac{1}{2} \sin 2\varphi \cdot \frac{1}{2} \cos 2\alpha \\ -\sqrt{2} \cdot \sin p \cdot \sin \varphi \cdot \frac{1}{2} \cos 2\alpha & -\sqrt{2} \cdot \cos p \cdot \frac{1}{2} \sin 2\varphi \cdot \frac{1}{2} \sin 2\alpha \\ -\frac{1}{\sqrt{2}} \sin p \cdot \cos \varphi \cdot \sin \alpha & +\sqrt{2} \cdot \cos p \cdot \frac{1}{2} \cos 2\varphi \cdot \cos \alpha \\ \frac{1}{\sqrt{2}} \sin p \cdot \cos \varphi \cdot \cos \alpha & +\sqrt{2} \cdot \cos p \cdot \frac{1}{2} \cos 2\varphi \cdot \sin \alpha \\ 0 & -\sqrt{\frac{3}{2}} \cos p \cdot \frac{1}{2} \sin 2\varphi \\ \frac{1}{\sqrt{2}} \sin p \cdot \cos \varphi \cdot \cos \alpha & +\frac{1}{\sqrt{2}} \cos p \cdot \sin \alpha \\ \frac{1}{\sqrt{2}} \sin p \cdot \cos \varphi \cdot \sin \alpha & -\frac{1}{\sqrt{2}} \cos p \cdot \cos \alpha \\ \frac{1}{\sqrt{2}} \sin p \cdot \sin \varphi \cdot \end{pmatrix} \quad (13),$$

C-vectors geometry in 9D space

Any unit \mathbf{C} -vector could be resolved into symmetrical \mathbf{C}_{sym} and antisymmetrical \mathbf{C}_{asym} components: $\mathbf{C} = \mathbf{C}_{sym} + \mathbf{C}_{asym}$. These components have the same length $|\mathbf{C}_{sym}| = |\mathbf{C}_{asym}| = 1/\sqrt{2}$ and are perpendicular to each other: $\mathbf{C}_{sym} \cdot \mathbf{C}_{asym} = 0$ (see Fig. 3). Both \mathbf{C} and \mathbf{C}_{sym} should be perpendicular to the \mathbf{T}_σ -vector we are looking for. The perpendicularity condition does not determine which sign of stress is correct, thus two numerical solutions \mathbf{T}_σ and $-\mathbf{T}_\sigma$ are possible, but only one of them produces the observed sense of movement (see Fig. 3).

To identify directly which sign is correct, we must focus on the antisymmetrical component \mathbf{C}_{asym} of \mathbf{C} -vector. It is easy to show that component \mathbf{C}_{asym} lies in asymmetrical subspace and is oriented in the same direction as the fault lineation. In other words, the sign of \mathbf{C}_{asym} indicates the sense of slip in real space:

$$C'_7 = \frac{l_x}{\sqrt{2}}; C'_8 = \frac{l_y}{\sqrt{2}}; C'_9 = \frac{l_z}{\sqrt{2}} \quad (14).$$

The \mathbf{C} -line could never be perpendicular to all of \mathbf{A}_{ij} vectors because it does not lie in symmetrical subspace (with \mathbf{C}_{sym}). Therefore the considered eight vectors ($\mathbf{A}_{xyz}, \mathbf{A}_{xy}, \mathbf{A}_{xz}, \mathbf{A}_{yz}, \mathbf{C}_I, \mathbf{C}_{II}, \mathbf{C}_{III}, \mathbf{C}_{IV}$) could never be perpendicular to each other. It means the value of the Gram determinant could never be equal to one. But the distance between the \mathbf{C} -vector and asymmetrical

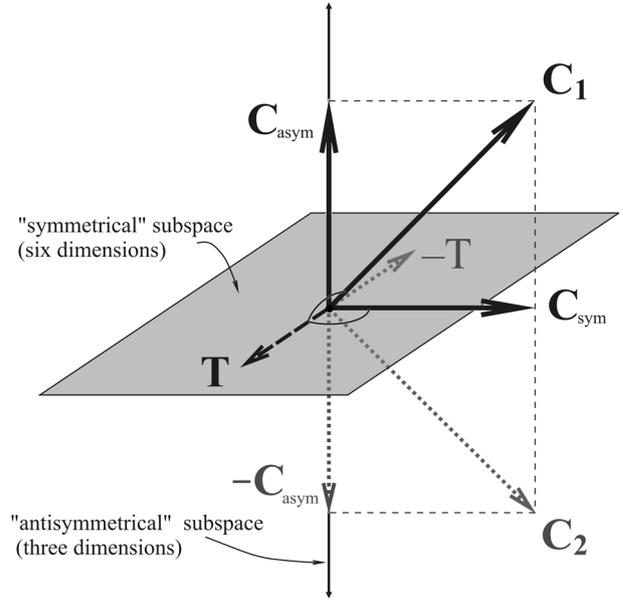


Figure 3. Symmetrical and antisymmetrical subspace of 9D space reduced to three dimensions. An antisymmetrical component \mathbf{C}_{asym} of the \mathbf{C} -vector lies in antisymmetrical subspace. On the other hand the symmetrical component \mathbf{C}_{sym} and the stress tensor vector \mathbf{T}_σ lie in symmetrical subspace. The opposite vectors are marked with a minus sign. Vector \mathbf{C}_2 represents the same fault and striation as vector \mathbf{C}_1 but with opposite sense of movement.

subspace (with \mathbf{C}_{asym}) is always 45 degrees, which constrains the maximum value of the square root of the Gram determinant \sqrt{G} to be equal to $1/8$, (i.e. 0.125).

So, what is the geometrical interpretation of \mathbf{C} -lines end points in symmetrical subspace? In fact, only three parameters (α, φ, p) determine a \mathbf{C} -line, therefore the end points of \mathbf{C} -lines create only 3-dimensional objects in 5 or 6D subspace, because explicit mapping into the 3D subspace could be defined. This complex spiral object can hardly be visualized. To illustrate this situation in only 3D space, we can imagine a helical line which is a 1D object (line), but dimensional reduction from 3D space brings about information loss. Another way is by excluding the α -parameter. It is obvious that coordinates C'_1 and C'_2 (C'_3 and C'_4 , respectively) represent similar expressions with differences in goniometric function of the α -parameter. New coordinates independent of the α -parameter, i.e. $C''_{1,2}$ and $C''_{3,4}$ can be expressed as a combination of two primed coordinates:

$$C''_{1,2} = \sqrt{C'^2_1 + C'^2_2} = \sqrt{\frac{1}{2} \sin^2 p \cdot \sin^2 \varphi + \frac{1}{8} \cos^2 p \cdot \sin^2 2\varphi} \quad (15),$$

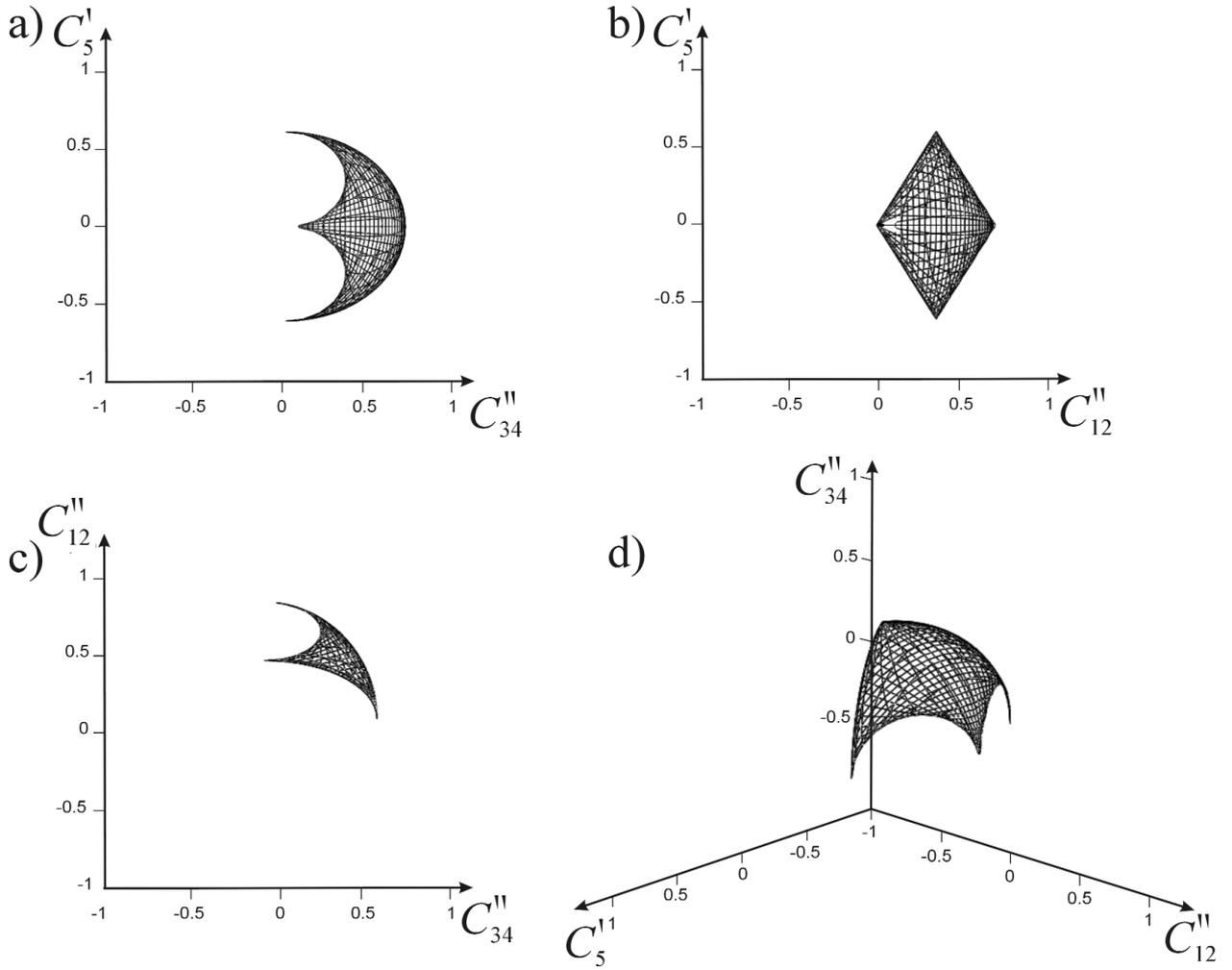


Figure 4. Distribution of end points of C-lines reduced into 3D space by exclusion of α -parameter. Two parameters remain (φ and p) representing 2D objects (part of the sphere surface) in 3D space with base coordinates $C_{1,2}''$, $C_{3,4}''$ and C_5' .

$$C_{3,4}'' = \sqrt{C_3'^2 + C_4'^2} = \sqrt{\frac{1}{2} \sin^2 p \cdot \cos^2 \varphi + \frac{1}{8} \cos^2 p \cdot \cos^2 2\varphi} \quad (16),$$

Accordingly, only two parameters continue to represent a 2D object (part of the sphere surface) in 3D space with base coordinates $C_{1,2}''$, $C_{3,4}''$ and C_5' (see Fig. 4).

Conclusions

This paper was focused on some details of paleostress analysis in 9D space:

1. A criterion based on the Gram determinant was proposed to evaluate the numerical output error.

2. The antisymmetrical component C_{asym} of a C-line corresponds to the direction and sense of striation; thus it may be used to recognize the correct sign of T_σ solution.

3. The symmetrical component C_{sym} of a C-line locally represents 3D objects in symmetrical 5D subspace.

Paleostress analysis of fault-slip data in 9D is a good way to illustrate the relationship between C-lines and the possible stress tensor solution.

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